

# Non-signalling theories and generalized probability

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We provide mathematically rigorous justification of using term *probability* in connection to the so called non-signalling theories, known also as Popescu's and Rohrlich's box worlds. No only do we prove correctness of these models (in the sense that they describe composite system of two independent subsystems) but we obtain new properties of non-signalling boxes and expose new tools for further investigation. Moreover, it allows straightforward generalization to more complicated systems.

## INTRODUCTION

The idea of so-called *non-signalling* theories, or *box worlds*, that originates from the paper of Popescu and Rohrlich [1], became a popular tool in certain areas related to quantum information theory, like proving security of ciphering protocols [2] or finding bounds on communication complexity [3] (to name only a few references out of many). By a *box world* one usually mean a physical model [4] build from two “black boxes”, where each box is characterized by a finite set of input values (that correspond to selection of an “observable”) and, for each input, finite set of output values. To fix notation, let us denote particular box world by  $(\mathcal{U}, \mathcal{V})$ , where  $\mathcal{U} = \{\mathcal{U}_1, \dots, \mathcal{U}_N\}$ ,  $\mathcal{V} = \{\mathcal{V}_1, \dots, \mathcal{V}_M\}$  and  $\mathcal{U}_i, \mathcal{V}_j$  are sets of outcomes for  $i$ -th input on left box and  $j$ -th input on right box, respectively.

A state of a box world is described with the help of the following thought experiment: we supply boxes with a large stream of input values and write down obtained outputs. Then we compute frequency  $P(\alpha\beta|ab)$  of getting outputs  $\alpha, \beta$  given the inputs  $a, b$  on the left and right box respectively [4].

Obviously,  $P$  must be a non-negative function and

$$\sum_{\alpha \in \mathcal{U}_a, \beta \in \mathcal{V}_b} P(\alpha\beta|ab) = 1, \quad a = 1, \dots, N, b = 1, \dots, M.$$

Moreover, we want boxes to be in some sense independent, thus we impose additional restrictions on  $P$ , namely the *non-signalling* conditions

$$\begin{aligned} \sum_{\alpha' \in \mathcal{U}_a} P(\alpha'\beta|ab) &= \sum_{\alpha' \in \mathcal{U}_a} P(\alpha'\beta|cb) \\ \sum_{\beta' \in \mathcal{V}_b} P(\alpha\beta'|ab) &= \sum_{\beta' \in \mathcal{V}_b} P(\alpha\beta'|ac) \end{aligned} \quad (1)$$

satisfied for any  $a, b, c, \alpha, \beta$ . These express the intuitive idea that the output of one of the boxes does not depend on the input of the other. It also reflects Einstein's causality principle in the model, as boxes could be placed in space-like separated regions of space-time.

It is assumed that any  $P$  that satisfies all these properties (non-negativity, normalization and non-signalling) is an admissible state for a given box world. In the sequel we will call such  $P$  a *PR-state* of the box world.

The problem with the above widely used formulation lies in the fact, that  $P$  is interpreted as a *probability* of getting output  $\alpha\beta$  given input  $ab$ . This interpretation is based on the thought experiment we discussed previously. We argue that applying frequentist definition of probability to thought experiments is unjustified. The number of “paradoxes” related to probability, like Bertrand's paradox, Loschmidt's paradox or Monty Hall problem to name only a few, shows that we should not rely solely on intuition when we talk about probability. On the other, hand without sound probabilistic interpretation of  $P$  the meaning of non-signalling conditions is questionable and so the whole idea of non-signalling theories.

Lacking of any physical realization of box worlds, we are forced to justify probabilistic interpretation of  $P$  on the basis of axiomatic approach to probability. Because box worlds obey neither classical nor quantum probability rules (cf. [5]) it is clear that we need framework that generalizes standard Kolmogorov's axioms of probability. Out of the two widely used such frameworks: operator algebra approach (that focused on the algebra of random variables) and quantum logic approach (that focused on a partially ordered set of random events), only the latter is capable of “super-quantum” generalization.

The paper is organized in the following way. We begin with a very brief introduction to quantum logics (detailed exposition can be found e.g. in [6]). Then we construct the logic of an arbitrary box world  $(\mathcal{U}, \mathcal{V})$ . We prove that in the framework of quantum logic it indeed can describe system composed of two independent subsystems and discuss some of its properties. We end the paper with some remarks about generalization of box worlds to larger number of boxes.

Finally we would like to mention that there is another mathematically rigorous attempt to formalize box world theories called General Probability Theory or Generic Probability Theory (GPT in short, see [7] and references

therein, although it seems that the idea appeared for the first time in 1974 in the Mielnik's paper [8]). The basic notion in the GPT is an arbitrary convex set of states. GPT is more general than the quantum logic framework that we use in this paper. On the other hand, the latter, due to its more restrictive nature, provides more tools to study features of the box world theories. Thus, our work can be considered as complementary to the convex set framework.

## QUANTUM LOGICS AS A FRAMEWORK FOR GENERALIZED PROBABILITY

Quantum logic approach originates from the seminal paper of Birkhoff and von Neumann [9]. A detailed physical introduction and justification of the whole programme can be found in the book of Piron [10], where the Hilbert space formulation of quantum mechanics is derived from the set of purely logical axioms. For us, it is important to note, that the notion of *probability* was always modeled over some physical system [11]. In case of classical probability, the sample space can be treated as a classical phase space (set of pure classical states, not necessarily Hamiltonian system), random events correspond to subsets of the phase space and probability measure is classical (in general mixed) state. So, by analogy, quantum probability can be defined by specifying the set random events, that correspond to orthogonal projectors on the Hilbert space of the system and a density matrix that defines a measure on the set of all projectors (we lack the notion of sample space, but that is why the quantum probability is quantum). These ideas motivate definitions presented in this section.

**Definition 1** A *quantum logic* is a partially ordered set  $\mathcal{L}$  with a map  $^\perp: \mathcal{L} \rightarrow \mathcal{L}$  such that

L1 there exists the greatest (denoted by  $\mathbb{1}$ ) and the least (denoted by  $0$ ) element in  $\mathcal{L}$ ,

L2 map  $p \mapsto p^\perp$  is order reversing, i.e.  $p \leq q$  implies that  $q^\perp \leq p^\perp$ ,

L3 map  $p \mapsto p^\perp$  is idempotent, i.e.  $(p^\perp)^\perp = p$ ,

L4 for a countable family  $\{p_i\}$ , s.t.  $p_i \leq p_j^\perp$  for  $i \neq j$ , the supremum  $\bigvee \{p_i\}$  exists,

L5 if  $p \leq q$  then  $q = p \vee (q \wedge p^\perp)$  (orthomodular law),

where  $p \vee q$  is the least upper bound and  $p \wedge q$  the greatest lower bound of  $p$  and  $q$ .

Elements of a quantum logic  $\mathcal{L}$  are interpreted as *propositions* about a physical system (equivalence class of experimental setups that result in one of two outcomes: either **true** or **false**). The partial order relation  $p \leq q$

is interpreted as “ $q$  is more plausible than  $p$ ” (e.g. whenever  $p$  is **true**,  $q$  is **true** as well). The greatest element of  $\mathcal{L}$  corresponds to trivial experimental questions, i.e. the ones that always result in **true**. Contrary, the least element corresponds to experimental questions that are always **false**. Map  $p \mapsto p^\perp$  encodes *negation*, i.e. whenever  $p$  is **true**,  $p^\perp$  is **false**, etc. Then L2 and L3 simply encode the basic properties of negation. Whenever  $p \leq q^\perp$  we say that  $p$  and  $q$  are *disjoint* and denote it by  $p \perp q$ . It is clear, that it means that  $p$  and  $q$  are mutually exclusive and consequently it should be meaningful to ask question “ $p$  or  $q$ ”. L4 extends this intuition to any countable family of disjoint elements. The last one, L5, lacks direct interpretation. One can think of it as a very form of weak distributivity. Nevertheless, it has profound technical importance.

If for any  $p, q \in \mathcal{L}$ ,  $p \vee q, p \wedge q$  exist, then  $\mathcal{L}$  is said to be an *orthomodular lattice*. On the other hand, the set of projectors on the Hilbert space, ordered by subspace inclusion and with  $P^\perp = I - P$  is always an orthomodular lattice. Thus we can define quantum probability as a orthomodular lattice. Moreover, if the distributivity law  $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$  holds, then  $\mathcal{L}$  is a Boolean algebra, i.e. one can find a set  $\Omega$  such that  $\mathcal{L}$  can be identified with a  $\sigma$ -algebra of its subsets.

Finally, let us point that it follows directly from L1 and L2 that both of de Morgan laws are satisfied, i.e. if  $p \vee q$  exists then  $p^\perp \wedge q^\perp$  exists and is equal to  $(p \vee q)^\perp$ , similarly if  $p \wedge q$  exists then  $p^\perp \vee q^\perp$  exists and is equal to  $(p \wedge q)^\perp$ .

**Definition 2** An element  $p \neq 0$  of a quantum logic  $\mathcal{L}$  is called an *atom* whenever  $0 \leq q \leq p$  implies that  $q = 0$  or  $q = p$ .  $\mathcal{L}$  is called *atomic* whenever for any  $q \in \mathcal{L}$  there exists an atom  $p \leq q$ . It is called *atomistic*, whenever any element  $q \in \mathcal{L}$  is a supremum of atoms less than  $q$ .

**Definition 3** A subset  $\mathcal{K} \subset \mathcal{L}$  of quantum logic  $\mathcal{L}$  is a *sublogic*, whenever (i) if  $p \in \mathcal{K}$  then  $p^\perp \in \mathcal{K}$  (ii) for any countable family  $\{p_i\} \subset \mathcal{K}$  of mutually disjoint elements  $\bigvee \{p_i\} \in \mathcal{K}$ .

**Definition 4** Let  $\mathcal{L}$  be a quantum logic. A *state*  $\rho$  on  $\mathcal{L}$  is a map  $\rho: \mathcal{L} \rightarrow [0, 1]$ , s.t.

S1  $\rho(\mathbb{1}) = 1$ ,

S2 for a countable family  $\{p_i\}$ , s.t.  $p_i \leq p_j^\perp$   $\rho(\bigvee \{p_i\}) = \sum_i \rho(p_i)$ .

Set of all states for quantum logic  $\mathcal{L}$  will be denoted by  $\mathcal{S}(\mathcal{L})$ .

Value  $\rho(p), p \in \mathcal{L}$  can be interpreted as probability of getting answer **true** for  $p$ . It is straightforward to check that this agrees with the interpretation of elements of  $\mathcal{L}$ . In particular, it can be easily shown, that if  $p \leq q$  then

$\rho(p) \leq \rho(q), \forall \rho$ . In physical applications, we usually assume that we have enough states to determine the order, i.e. if  $\rho(p) \leq \rho(q), \forall \rho$  then  $p \leq q$ .

From the physical point of view we are often interested in subsets of observables which outcomes can be described by classical probability models (e.g. complete set of commuting observables in quantum physics, or in other words, simultaneously measureable observables). For this purpose we define:

**Definition 5** Let  $\mathcal{L}$  be a quantum logic. Then  $p, q \in \mathcal{L}$  are *compatible*, what we denote by  $p \leftrightarrow q$ , whenever there exist pairwise disjoint questions  $p_1, q_1, r$  such that  $p = p_1 \vee r, q = q_1 \vee r$ .

More generally, a subset  $A \subset \mathcal{L}$  is said to be *compatible* whenever for any finite subset  $\{p_1, \dots, p_n\} \subset A$  there exist finite subset  $G \subset \mathcal{L}$ , such that (i) elements of  $G$  are mutually disjoint, (ii) any  $p_i$  is supremum of some subset of  $G$ .

This definition looks very technical. It is easier to think of compatibility in terms of the following property:

**Theorem 6 ([12], Thm. 1.3.23)** *Let  $A \subset \mathcal{L}$  be a compatible subset of quantum logic. Then there exists a Boolean sublogic  $\mathcal{K} \subset \mathcal{L}$ , s.t.  $A \subset \mathcal{K}$ .*

Observe, that in general pairwise compatibility in  $A \subset \mathcal{L}$  is not sufficient for existence of a Boolean sublogic containing  $A$  (cf. [12] for examples)

Although elements of  $\mathcal{L}$  can be abstract, we will be interested in one particular class of quantum logics, namely:

**Definition 7 (see [12], Sec. 1.1)** A *concrete logic*  $\Delta$  is a family of subsets of some set  $\Omega$  with partial order relation given by set inclusion and  $A^\perp = \Omega \setminus A$  satisfying:

$$\text{C1 } \emptyset \in \Delta,$$

$$\text{C2 } A \in \Delta \text{ implies } \Omega \setminus A \in \Delta,$$

$$\text{C3 for any countable family } \{A_i\} \subset \Delta \text{ of mutually disjoint sets } \bigcup \{A_i\} \in \Delta.$$

**Example 8** Let  $\Omega = \{1, \dots, 2k\}$  and  $\Delta$  be a family of subsets  $\Omega$  with even number of elements. Then  $\Delta$  is a concrete logic, which is a Boolean algebra for  $k = 1$ , an orthomodular lattice for  $k = 2$  and a quantum logic for  $k = 3$ . ■

Although concrete logics are obviously more general than Boolean algebras and even orthomodular lattices, they exhibit some classical properties. For example, Heisenberg uncertainty relations are not satisfied in concrete logics [6], precisely

**Definition 9** An *observable*  $X$  on a quantum logic  $\mathcal{L}$  is a map  $X: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}$ , where  $\mathcal{B}(\mathbb{R})$  is a set of Borel subsets of real line, such that:

$$\text{O1 } X(\mathbb{R}) = \mathbb{1},$$

$$\text{O2 } X(\mathbb{R} \setminus A) = X(A)^\perp,$$

$$\text{O3 } X(A_1 \cup \dots \cup A_n) = X(A_1) \vee \dots \vee X(A_n) \text{ for any family of mutually disjoint } A_i\text{'s.}$$

For an observable  $X$  we can define its expected value and variance by

$$\mu(X) := \int_{\mathbb{R}} t \mu(X(dt)),$$

$$\Delta_\mu X := \int_{\mathbb{R}} (t - \mu(X))^2 \mu(X(dt)),$$

whenever integrals exist. Then for any pair of observables  $X, Y$  with finite expected value and variance one of two conditions is satisfied, either

$$\forall \varepsilon > 0 \exists \text{ a state } \mu \text{ with finite variance for } X \text{ and } Y,$$

$$(\Delta_\mu X)(\Delta_\mu Y) < \varepsilon \quad (2)$$

or

$$\exists \varepsilon > 0 \forall \text{ states } \mu \text{ with finite variance for } X \text{ and } Y,$$

$$(\Delta_\mu X)(\Delta_\mu Y) \geq \varepsilon. \quad (3)$$

In the former case we say that *Heisenberg uncertainty relations are not satisfied*, while in the latter case we say that *Heisenberg uncertainty relations are satisfied*. Then it follows that if  $\mathcal{L}$  is a concrete logic then the Heisenberg uncertainty relations are not satisfied (see Thm. 50 and Thm. 129 in [6]).

## CONCRETE LOGIC OF NON-SIGNALLING THEORIES

Let us fix our attention on arbitrary box world  $(\mathcal{U}, \mathcal{V})$ , with  $N, M$  be the number of distinct input values on left box and right box respectively. We define sets

$$\Gamma_1 = \{(x_1, x_2, \dots, x_N) \mid x_i \in \mathcal{U}_i\},$$

$$\Gamma_2 = \{(y_1, y_2, \dots, y_M) \mid y_i \in \mathcal{V}_i\},$$

$$\Gamma = \Gamma_1 \times \Gamma_2$$

If boxes obeyed classical physics, then  $\Gamma_1, \Gamma_2, \Gamma$  would correspond to the phase spaces of left box, right box and the phase space of a composite system.

With any experimental question of the form:

$$\text{“does input } a \text{ on left box and } b \text{ on right result in output } \alpha \text{ on left box and } \beta \text{ on right?”} \quad (4)$$

we assign an element  $[a\alpha, b\beta] \subset \Gamma$  in the following way:

$$[a\alpha, b\beta] = \{(x, y) \in \Gamma \mid x_a = \alpha, y_b = \beta\} \quad (5)$$

We denote by  $\mathcal{A}$  the set of all such subsets of  $\Gamma$ . Let  $\mathcal{L}$  be a sublogic of Boolean algebra  $2^\Gamma$  generated by  $\mathcal{A}$  (the smallest sublogic containing  $\mathcal{A}$ ; it exists, see [12]). It follows that:

**Observation 10**  $\mathcal{L}$  is a concrete logic with finite number of elements. Moreover, any element of  $\mathcal{L}$  is either a finite union of mutually disjoint sets from  $\mathcal{A}$  or is an empty set.

Let us observe that  $[a\alpha, b\beta] \perp [c\gamma, d\delta]$  whenever  $[a\alpha, b\beta] \cap [c\gamma, d\delta] = \emptyset$ , i.e. when (i)  $a = c$  and  $\alpha \neq \gamma$  or (ii)  $b = d$  and  $\beta \neq \delta$ . We will write  $p \oplus q$  to indicate  $p \vee q$  with implicit assumption that  $p \perp q$ .

**Lemma 11** If  $p = [a\alpha, b\beta] \leq q$ , then either (i)  $q = p \oplus q'$ , (ii)  $q = [a\alpha, \mathbb{1}] \oplus q'$ , (iii)  $q = [\mathbb{1}, b\beta] \oplus q'$  or (iv)  $q = \mathbb{1}$ , where

$$\begin{aligned} [a\alpha, \mathbb{1}] &= \{(x, y) \in \Gamma \mid x_a = \alpha\}, \\ [\mathbb{1}, b\beta] &= \{(x, y) \in \Gamma \mid y_b = \beta\}. \end{aligned}$$

*Proof* Let  $q = q_1 \oplus \dots \oplus q_n$  with  $q_i = [a_i\alpha_i, b_i\beta_i] \in \mathcal{A}$ . Alternative (i) is obvious, so let us assume that none of  $q_i$ 's equals  $p$ . Let first  $k$  of  $q_i$ 's be not disjoint with  $p$ , i.e.  $p \not\perp q_i$  for  $i = 1, \dots, k$  and  $p \perp q_i$  for  $i = k+1, \dots, n$ . Thus any of  $q_i$ 's for  $i = 1, \dots, k$  must be of the form: (I)  $q_i = [a_i\alpha_i, b\beta]$ , (II)  $q_i = [a\alpha, b_i\beta_i]$  or (III)  $q_i = [a_i\alpha_i, b_i\beta_i]$ , where  $a_i \neq a$ ,  $b_i \neq b$ .

Let us firstly examine the case when one of  $q_i$ 's, say  $q_1$ , is of the form (I). By mutual disjointness of  $q_i$ 's all remaining  $i = 2, \dots, k$  are either of form (I) or (III). Let  $\alpha' \in \mathcal{U}_{a_1}$ ,  $\alpha' \neq \alpha_1$  be another output for input  $a_1$ . Then  $(x, y) \in p$ , where  $x_a = \alpha$ ,  $x_{a_1} = \alpha'$ ,  $y_b = \beta$  and rest arbitrary, but  $(x, y) \notin q_1$ . Consequently, there must be another  $q_i$ , say  $q_2$ , such that  $(x, y) \in q_2$ . By previous comment, either (a)  $q_2 = [a_1\alpha', b\beta]$  (form (I) disjoint with  $q_1$ ) or (b)  $q_2 = [a_1\alpha', b_2\beta_2]$  (form (III) disjoint with  $q_1$ ).

Assume that (b) is the case. Take another  $(x', y') \in p$  equal to  $(x, y)$  except for  $y_{b_2} = \beta' \neq \beta_2$ . Then  $(x', y') \notin q_1$  since  $x_{a_1} \neq \alpha_1$  and  $(x', y') \notin q_2$  since  $y_{b_2} \neq \beta_2$ . Again, one of  $q_i$ 's must be of the form  $q_i = [a_1\alpha', b_2\beta']$  (form (III) disjoint with  $q_1$  and  $q_2$ ). Observe that there is no admissible form (I) disjoint with  $q_1$  and  $q_2$  for that case. If we repeat this reasoning for all outputs  $\mathcal{V}_{b_2}$  we conclude that there is some set  $I$  of  $i = 1, \dots, k$  such that  $\bigoplus_{i \in I} q_i = [a_1\alpha', \mathbb{1}]$ . But  $[a_1\alpha', \mathbb{1}] = [a_1\alpha', b\beta] \oplus r$ , where  $r \perp p$ , thus case (b) reduces to (a).

Consequently, we can assume without loss of generality (a) is always the case. We repeat reasoning for all outputs in  $\mathcal{U}_{a_1}$  and conclude that there must be some set  $J$  of  $i = 1, \dots, k$  such that  $\bigoplus_{i \in J} q_i = [\mathbb{1}, b\beta]$ . The case when one of  $q_i$ 's is of the form II is symmetric.

Finally, let  $q_1 = [a_1\alpha_1, b_1\beta_1]$ . Take  $(x, y) \in p$  such that  $x_a = \alpha$ ,  $x_{a_1} = \alpha'$ ,  $y_b = \beta$ ,  $y_{b_1} = \beta_1$  and rest arbitrary. Clearly  $(x, y) \notin q_1$ . Let  $(x, y) \in q_2$ . Since all of  $q_i$ 's must be of the form III,  $q_2 \perp q_1$  result in either (a)  $q_2 = [a_1\alpha', b_2\beta_2]$ ,  $b_2 \neq b_1$  or (b)  $q_2 = [a_1\alpha', b_1\beta_1]$ .

Assume (a). Take another  $(x', y') \in p$  equal to  $(x, y)$  except for  $y'_{b_2} = \beta'$ . Then there must be  $q_i$ , such that  $(x', y') \in q_i$  and  $q_i \perp q_1$  and  $q_i \perp q_2$  require that

$q_i = [a_1\alpha', b_2\beta']$ . Like previously, we repeat this for all possible outcomes of  $b_2$  and conclude that there must be subset  $I$  of  $i = 1, \dots, k$  such that  $\bigoplus_{i \in I} q_i = [a_1\alpha_1, \mathbb{1}] = [a_1\alpha_1, b\beta] \oplus r$ , so we can rewrite  $q$  in the way that there is  $q_i$  of the form (I).

On the other hand, if (b) is the case, we repeat reasoning for all other outputs of  $a_1$  and then symmetrically for all outputs of  $b_1$  and conclude that  $q = \mathbb{1}$ . ■

**Theorem 12** Let  $P$  be a PR-box state on  $(\mathcal{U}, \mathcal{V})$ -box world and let  $\mathcal{L}$  be an above defined logic. Then  $\rho_P: \mathcal{L} \rightarrow [0, 1]$  defined by

$$(i) \rho_P(\emptyset) = 0,$$

$$(ii) \rho_P([a\alpha, b\beta]) = P(\alpha\beta|ab),$$

$$(iii) \rho_P(\bigvee_i p_i) = \sum_i \rho_P(p_i), \text{ for any set } \{p_i\} \subset \mathcal{A} \text{ of pairwise disjoint elements}$$

is a state on  $\mathcal{L}$ . On the other hand, any state  $\rho$  on  $\mathcal{L}$  defines a PR-state  $P_\rho$  on  $(\mathcal{U}, \mathcal{V})$ -box world by:  $P_\rho(\alpha\beta|ab) = \rho([a\alpha, b\beta])$ .

*Proof* We need to show that the definition of  $\rho_P$  is correct, i.e. for all  $p \in \mathcal{L}$ , and for all decompositions  $p = p_1 \oplus \dots \oplus p_n$  (iii) results in the same value. Then the fact that  $\rho$  is a state follows directly from its definition.

The simplest case  $p = \mathbb{1}$  follows from normalization of PR-box state. If  $p = [a\alpha, \mathbb{1}]$  or  $p = [\mathbb{1}, b\beta]$  for some  $a, \alpha, b, \beta$  then decomposition is not unique, but non-signalling condition guarantess that (iii) does not introduce any ambiguity. It follows from lemma 11 that  $p$  has non-unique decomposition into atoms only if  $[a\alpha, \mathbb{1}] \leq p$  or  $[\mathbb{1}, b\beta] \leq p$  for some  $a, \alpha, b, \beta$ . Consequently, any  $p$  can be decomposed into:  $p = p_1 \oplus \dots \oplus p_k$  where  $p_i$  has one of three forms: (A)  $p_i = [a_i\alpha_i, b_i\beta_i]$ , (B)  $p_i = [a_i\alpha_i, \mathbb{1}]$  or (C)  $p_i = [\mathbb{1}, b_i\beta_i]$ .

Consequently, it remains to show that (iii) is valid in case of  $[a\alpha, \mathbb{1}] \oplus p = [\mathbb{1}, b\beta] \oplus q$ . Let  $q = q_1 \oplus \dots \oplus q_k$  be decomposition into atoms. Observe that for all  $(x, y) \in p$ , with  $y_b = \beta' \neq \beta$   $(x, y) \notin [\mathbb{1}, b\beta]$ , so for any  $\beta' \in \mathcal{V}_b \setminus \beta$  there is (possible more than one)  $q_i = [c_i\gamma_i, b\beta']$ .

Let  $q_1 = [c\gamma, b\beta_1]$  and assume that  $c_1 \neq a$ . Take  $(x, y) \in [a\alpha, \mathbb{1}]$ , such that  $x_a = \alpha$ ,  $x_c = \gamma' \neq \gamma$ ,  $y_b = \beta_1$ . Then  $(x, y) \notin [\mathbb{1}, b\beta]$  and  $(x, y) \notin q_1$ . There must be  $q_i = [c_i\gamma_i, b\beta_i]$  such that  $(x, y) \in q_i$ , but  $q_i \perp q_1$  requires that  $c_i = c$ ,  $\gamma_i = \gamma'$ . Repeating this reasoning we conclude that there must be subset  $I$  of  $i = 1, \dots, k$  such that  $\bigoplus_{i \in I} q_i = [\mathbb{1}, b\beta_1]$ . On the other hand, if  $c_1 = a$ , then we immediately get that  $q_1 = [a\alpha, b\beta_1]$ . Consequently,  $q$  can be written in the form  $q = \bigoplus_{\beta' \in \mathcal{V}_b^1} [a\alpha, b\beta'] \oplus \bigoplus_{\beta' \in \mathcal{V}_b^2} [\mathbb{1}, b\beta']$  where  $\mathcal{V}_b^1 \cup \mathcal{V}_b^2 = \mathcal{V}_b \setminus \beta$ . The same can be done for  $p$ . Now it is easy to observe that by expanding terms  $[a\alpha_i, \mathbb{1}]$  and  $[\mathbb{1}, b\beta_i]$  we can obtain the same atomic decomposition on the left and right. By transitivity of equality relation, we conclude that (iii) is well defined.



The second part of theorem follows immediately from the fact, that non-signaling condition actually means that there are elements of the type  $[a\alpha, \mathbb{1}], [\mathbb{1}, b\beta]$ :

$$\sum_{\beta \in \mathcal{V}_b} \rho([a\alpha, b\beta]) = \rho([a\alpha, \mathbb{1}]) = \sum_{\gamma \in \mathcal{V}_c} \rho([a\alpha, c\gamma]), \quad \text{etc.} \quad \blacksquare$$

Now we are going to argue that  $\mathcal{L}$  is the logic of  $(\mathcal{U}, \mathcal{V})$ -box world. Firstly, let us observe that the set of states on  $\mathcal{L}$  is order determining. On the other hand, in the operational construction of the logic of the physical system one *assumes* that the set of physical states determines the order. Consequently, states of the logic of  $(\mathcal{U}, \mathcal{V})$ -box world induced by the PR-states should be order determining. Secondly, elements of  $\mathcal{A}$  clearly correspond to atoms of the  $(\mathcal{U}, \mathcal{V})$ -box world logic. Moreover, definition of the  $(\mathcal{U}, \mathcal{V})$ -box world explicitly enumerates all possible the most elementary experiment that we can perform on the box world. This somehow forces us to assume, that the logic of  $(\mathcal{U}, \mathcal{V})$ -box world is not only atomic, but also atomistic. Summarizing, the logic  $\mathcal{L}$  is the minimal completion of the set  $\mathcal{A}$  that has sufficiently many states (all PR-box states are represented) and is consistent with definition of box world (all states are non-signaling).

Although the definition of  $\mathcal{L}$  as a certain concrete logic modeled on a classical system seems to be a lucky guess, it emerged from the study of properties the abstractly constructed logic of exemplary box world system with binary input and binary output for both boxes [13].

Elements  $[a\alpha, \mathbb{1}], [\mathbb{1}, b\beta]$  (and their valid  $\oplus$ -sums) can be interpreted as propositions about only one of boxes, thus we call them *localized in A* or *localized in B* respectively. Observe, that  $[a\alpha, \mathbb{1}] \perp [a'\alpha', \mathbb{1}]$  if and only if  $a = a'$  and  $\alpha \neq \alpha'$ . We will use following notation:

$$[a \in \mathcal{P}, \mathbb{1}] = \oplus\{[a\alpha, \mathbb{1}] \mid \alpha \in \mathcal{P}\}, \text{ where } \mathcal{P} \subset \mathcal{U}_a.$$

**Proposition 13** *Any pair of elements  $p, q \in \mathcal{L}$  such that  $p$  is localized in A and  $q$  is localized in B is compatible.*

*Proof* Let  $p = [a \in \mathcal{P}, \mathbb{1}]$ ,  $q = [\mathbb{1}, b \in \mathcal{Q}]$ . Let us define  $p_1 = \oplus_{\alpha \in \mathcal{P}, \beta \in \mathcal{V}_b \setminus \mathcal{Q}} [a\alpha, b\beta]$ ,  $q_1 = \oplus_{\alpha \in \mathcal{U}_a \setminus \mathcal{P}, \beta \in \mathcal{Q}} [a\alpha, b\beta]$ ,  $r = \oplus_{\alpha \in \mathcal{P}, \beta \in \mathcal{Q}} [a\alpha, b\beta]$ . It is clear that  $p_1, q_1, r$  are mutually disjoint and  $p = p_1 \oplus r$ ,  $q = q_1 \oplus r$ .  $\blacksquare$

**Lemma 14** *Let  $p = [a \in \mathcal{P}, \mathbb{1}]$ ,  $q = [a' \in \mathcal{Q}, \mathbb{1}]$ . Then:*

$$p \wedge q = \begin{cases} 0 & \text{if } a \neq a', \\ [a \in \mathcal{P} \cap \mathcal{Q}, \mathbb{1}] & \text{otherwise,} \end{cases}$$

$$p \vee q = \begin{cases} \mathbb{1} & \text{if } a \neq a', \\ [a \in \mathcal{P} \cup \mathcal{Q}, \mathbb{1}] & \text{otherwise.} \end{cases}$$

*Proof* Firstly, let us assume that  $a \neq a'$ . Then  $p \wedge q$  must be a subset of

$$p \cap q = \{(x, y) \mid x_a \in \mathcal{P}, x_{a'} \in \mathcal{Q}\}$$

but this set has constraints on two elements of  $x$ , so there is no non empty  $r \in \mathcal{L}$ , such that  $r \subset p \cap q$ . Consequently,  $p \wedge q = 0$ . Then  $p \vee q = \mathbb{1}$  follows from the de Morgan law. When  $a = a'$  the proof is obvious ( $p \cap q = p \wedge q$  and  $p \cup q = p \vee q$ ).  $\blacksquare$

**Proposition 15** *Elements  $p = [a \in \mathcal{P}, \mathbb{1}]$ ,  $q = [a' \in \mathcal{Q}, \mathbb{1}]$  are compatible if and only if  $a = a'$ .*

*Proof* We will show the non-trivial implication by contraposition. Assume that  $a \neq a'$ . Since  $p^\perp = [a \in \mathcal{U}_a \setminus \mathcal{P}, \mathbb{1}]$

$$p \vee (p^\perp \wedge q) = p \vee 0 = p,$$

and

$$(p \vee p^\perp) \wedge (p \vee q) = \mathbb{1} \wedge \mathbb{1} = \mathbb{1},$$

so  $p, p^\perp, q$  cannot be contained in Boolean sublogic of  $\mathcal{L}$ .  $\blacksquare$

**Theorem 16** *A set of pairwise compatible localized elements  $[a \in \mathcal{P}_1, \mathbb{1}], \dots, [a \in \mathcal{P}_k, \mathbb{1}], [\mathbb{1}, b \in \mathcal{Q}_1], \dots, [\mathbb{1}, b \in \mathcal{Q}_l]$  is compatible.*

*Proof* Denote  $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$  and  $\mathcal{Q} = \bigcup_{j=1}^l \mathcal{Q}_j$ .

Clearly there is a mutually disjoint partition  $\{\tilde{\mathcal{P}}_s\}_{s=1}^K$  of  $\mathcal{P}$ , such that any  $\mathcal{P}_i = \bigcup_{s \in I_i} \tilde{\mathcal{P}}_s$  for some  $I_i \subset \{1 \dots K\}$ . Similarly, there is a mutually disjoint partition  $\{\tilde{\mathcal{Q}}_s\}_{s=1}^L$  of  $\mathcal{Q}$  such that any  $\mathcal{Q}_j = \bigcup_{s \in J_j} \tilde{\mathcal{Q}}_s$  for some  $J_j \subset \{1 \dots L\}$ . Let us define:

$$p_i = \bigoplus\{[a\alpha, b\beta] \mid \alpha \in \tilde{\mathcal{P}}_i, \beta \in \mathcal{V}_b \setminus \mathcal{Q}\},$$

$$q_j = \bigoplus\{[a\alpha, b\beta] \mid \alpha \in \mathcal{U}_a \setminus \mathcal{P}, \beta \in \tilde{\mathcal{Q}}_j\},$$

$$r_{ij} = \bigoplus\{[a\alpha, b\beta] \mid \alpha \in \tilde{\mathcal{P}}_i, \beta \in \tilde{\mathcal{Q}}_j\}.$$

Clearly  $p_i, q_j, r_{ij}$  are mutually disjoint. Moreover

$$[a \in \mathcal{P}_i, \mathbb{1}] = \bigoplus_{s \in I_i} \left( p_s \oplus \bigoplus_{j=1}^l \bigoplus_{t \in J_j} r_{st} \right),$$

$$[\mathbb{1}, b \in \mathcal{Q}_j] = \bigoplus_{t \in J_j} \left( q_t \oplus \bigoplus_{i=1}^k \bigoplus_{s \in I_i} r_{st} \right). \quad \blacksquare$$

The last Theorem is crucial for the interpretation of  $(\mathcal{U}, \mathcal{V})$ -box world system as a system composed of two separate subsystems. As was noted in the proof of Thm. 12 non-signaling condition is statement of merely existence of certain elements of the logic. It is far from obvious that this implies any form of compatibility in general (although we do not rule out such possibility). It is worth to mention here, that Coecke points out that in the context of process theory the non-signaling condition is also not the most adequate notion [14].

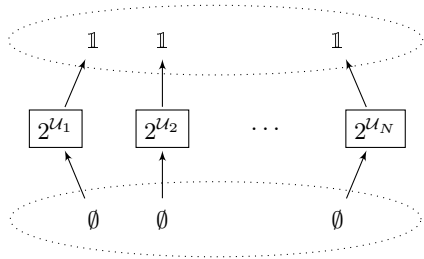


FIG. 1. The 0, 1-pasting of Boolean algebras  $\mathcal{B}_a = 2^{\mathcal{U}_a}$ , i.e. disjoint sum of all Boolean blocks  $\mathcal{B}_a$ , with minimal and maximal elements identified.

**Corollary 17** *The logic of a single box in an  $(\mathcal{U}, \mathcal{V})$ -box world is a 0, 1-pasting of Boolean logics  $\mathcal{B}_a = 2^{\mathcal{U}_a}$ , i.e. the logic  $L_1$  of left box is a disjoint union of  $\{\mathcal{B}_a\}_{a=1}^N$  modulo by equivalence relation that identify 0's and 1's of all  $\mathcal{B}_a$ 's (cf. Figure 1). Consequently, the logic of single box is an orthomodular lattice.*

*Proof* It is clear that  $\mathcal{B}_a = \{[a \in \mathcal{P}, 1] \mid \mathcal{P} \subset \mathcal{U}_a\}$  is a sublogic of  $\mathcal{L}$ , which is a Boolean logic. Then the claim follows directly from the Lemma 14. ■

We would like to emphasize that this shows that the logic of single boxes in box-world models is naively simple. It consists of the Boolean logics glued together in the most “free” way. There are no relations imposed between observables defined by different inputs.

## CONCLUSIONS

We constructed propositional system for a box world model consisting of two boxes with any finite number of inputs and outputs. Obtained structure justifies probabilistic interpretation of box world models in the sense of quantum logics. Our construction extends in an obvious way to many box models, although their properties were not yet studied by us.

Quantum logic approach allows for a more refined examination of the model that the convex set approach (in the sense of [7]). In particular, we were able to discuss notion of compatibility, Heisenberg uncertainty relations and we identified single box logics.

Our results allow to conclude that, contrary to the common belief (cf. [15],[4]) box world theories cannot be considered as a more general than quantum theory, even if one restricts to unphysical finite dimensional Hilbert spaces. The reason is three-fold:

1. logic of box-world model has always finite number of elements, while the logic of even the simplest quantum model (two level system) has an infinite number of elements,
2. logic of single boxes, being a 0, 1-pasting of Boolean algebras, is structurally much simpler than the propositional systems of quantum mechanics,
3. the logic of box-world model is set-representable and consequently such models do not satisfy Heisenberg uncertainty relations.

Although such models permit stronger correlations than possible in quantum mechanics, we emphasize that this is only one of many aspects of probability theory.

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